Pergamon

Int. J. Solids Structures Vol. 35, No. 13, pp. 1385–1394, 1998

0020–7683/98 $19.00 + .00

PII: S0020–7683(97)0011–X

BOUNDARY ELEMENT ANALYSIS OF THE STRESS SINGULARITY AT THE INTERFACE CORNER OF VISCOELASTIC ADHESIVE LAYERS

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(*Received* 17 *September* 1996; *in revised form* 25 *March* 1997)

Abstract—This paper concerns the stress singularity at the interface corner between the adhesive layer and the rigid adherend subjected to a uniform transverse tensile strain. The adhesive is assumed to be a linear viscoelastic material. The standard Laplace transform technique is employed to get the characteristic equation and the order of the singularity is obtained numerically for given viscoelastic models. The time-domain boundary element method is used to investigate the behavior of stresses for the whole interface. For the viscoelastic models considered, it is shown that the free-edge stress intensity factors are relaxed with time, while the order of the singularity increases with time or remains constant. © 1998 Elsevier Science Ltd.

1. INTRODUCTION

The problem of bonded quarter planes consisting of two isotropic and elastic materials has received much attention (Bogy, 1968; Reedy, 1990; Tsai and Morton, 1991). It was shown by Bogy (1968) that a stress singularity of type r^{δ} exists at the interface corner between bonded elastic quarter planes and δ is the solution of a characteristic equation. The order of the singularity at the free edge of the interface is dependent on the elastic constants of the two materials.

The interface of adhesively bonded materials would suffer from a stress system in the vicinity of the free surface under a transverse tensile loading. In such a region two interacting free surface effects occur, and very large interface stresses can be produced. A stress singularity which exists at the interface corner between the adherend and the adhesive layer might lead to adherend–adhesive debonding. In this study, the stress singularity at the interface corner between the rigid adherend and the viscoelastic adhesive subjected to a uniform transverse tensile strain is investigated. At room temperature the adhesive remains in its initial glassy stage through the entire loading period and, hence, it is not necessary to consider the time-dependent behavior of the stress–strain relationships in performing the stress analysis of bonded materials. In certain application, however, the temperature and time dependence of the loading may be such that the rheological behavior of the adhesive materials may no longer be negligible. Hence, provision is made for the adhesive assumed to be a linear viscoelastic material.

The interface stresses in viscoelastic adhesive layers have been studied by several investigators. Weitsman (1979) considered a pair of interfaces in which the adhesive material is viscoelastic and the adherend is rigid. Delale and Erdogan (1981) analyzed an adhesively bonded lap joint by assuming that one material is elastic and the other is viscoelastic. The results exhibited a redistribution of the very large stresses near the edge of the interface, but no singularities were encountered because of the simplifying assumptions with regard to the modeling of joining structural members.

In this study, the transformed characteristic equation for perfectly bonded rigid adherend and viscoelastic adhesive materials is first derived, following Williams (1952), with the use of the Laplace transform with respect to time t. This equation is inverted analytically for given viscoelastic models into the time-dependent viscoelastic equation which is readily solved using standard numerical procedure. The time-domain boundary element method (BEM) is then employed to investigate the behavior of stresses at the interface of viscoelastic adhesive layers subjected to a uniform transverse tensile strain.

2. ORDER OF THE SINGULARITY AT THE INTERFACE CORNER

The region near the interface corner between perfectly bonded viscoelastic and rigid quarter planes is shown in Fig. 1. In the following, a condition of plane strain is considered. A solution of

$$\nabla^4 \phi(r, \theta; t) = 0 \tag{1}$$

is to be found such that the normal stress, $\sigma_{\theta\theta}$, and shear stress, $\tau_{r\theta}$, vanish along $\theta = -\pi/2$, further that the displacements are zero across the common interface line $\theta = 0$. The solution of this problem is facilitated by the Laplace transform, defined as

$$\phi^*(r, \theta; s) = \int_0^\infty \phi(r, \theta; t) e^{-st} \, dt \tag{2}$$

where ϕ^* denotes the Laplace transform of ϕ and s is the transform parameter. Then eqn (1) can be rewritten using eqn (2) as follows:

$$\nabla^4 \phi^*(r, \theta; s) = 0. \tag{3}$$

By definition, the viscoelastic stresses in the Laplace transformed space are found from the stress function ϕ^* in the following manner:

$$\begin{aligned} \sigma^*_{rr} &= \frac{1}{r}\phi^*_{,r} + \frac{1}{r^2}\phi^*_{,\theta\theta} \\ \sigma^*_{\theta\theta} &= \phi^*_{,rr} \\ \tau^*_{r\theta} &= \frac{1}{r^2}\phi^*_{,\theta} - \frac{1}{r}\phi^*_{,r\theta} \end{aligned} \tag{4}$$

and the strains can be shown to be given by

$$\begin{aligned} u^*_{r,r} &= \frac{1}{2s\mu^*(s)}\left[\frac{1}{r}\phi^*_{,r} + \frac{1}{r^2}\phi^*_{,\theta\theta} - s\nu^*(s)\nabla^2\phi^*\right] \\ u^*_{\theta,r} - \frac{u^*_\theta}{r} + \frac{u^*_{r,\theta}}{r} &= \frac{1}{s\mu^*(s)}\left[\frac{1}{r^2}\phi^*_{,\theta} - \frac{1}{r}\phi^*_{,r\theta}\right] \end{aligned} \tag{5}$$

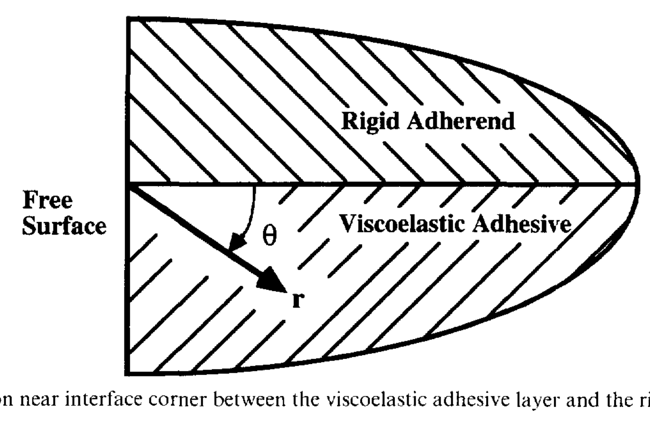

Fig. 1. Region near interface corner between the viscoelastic adhesive layer and the rigid adherend.

where σ_{ij}^* and u_i^* are the Laplace transformed stresses and displacements, respectively, and μ^* and v^* are Laplace transforms of the shear relaxation modulus $\mu(t)$ and the viscoelastic Poisson's ratio $v(t)$. Combining these equations with the traction-free boundary conditions (at $\theta = -\pi/2$)

$$\sigma_{\theta\theta}^* = \tau_{r\theta}^* = 0 \tag{6}$$

and the interface conditions (at $\theta = 0$)

$$u_r^* = u_\theta^* = 0 \tag{7}$$

one can solve the problem.

Using a method similar to that described in Williams (1952), a stress function of the form

$$\phi^*(r,\theta;s) = r^{\lambda+1}f(\theta;s), \ -\pi/2 \leqslant \theta \leqslant 0, \quad r > 0 \tag{8}$$

is assumed, where r and θ are defined in Fig. 1. Typical solution for $f(\theta;s)$ is chosen of the form

$$f(\theta;s) = c_1(s)\sin(\lambda+1)\theta + c_2(s)\cos(\lambda+1)\theta + c_3(s)\sin(\lambda-1)\theta + c_4(s)\cos(\lambda-1)\theta \tag{9}$$

where c_i are arbitrary constants. Then, stresses and displacements are given as follows:

$$\begin{aligned}
\sigma_{rr}^* &= r^{\lambda-1}\left[\frac{d^2}{d\theta^2}f(\theta;s) + (\lambda+1)f(\theta;s)\right] \\
\sigma_{\theta\theta}^* &= r^{\lambda-1}(\lambda+1)\lambda f(\theta;s) \\
\tau_{r\theta}^* &= -r^{\lambda-1}\lambda\frac{d}{d\theta}f(\theta;s) \\
u_r^* &= \frac{r^\lambda}{2s\mu^*(s)}\left[-(\lambda+1)f(\theta;s) + [1-sv^*(s)]\frac{d}{d\theta}g(\theta;s)\right] \\
u_\theta^* &= \frac{r^\lambda}{2s\mu^*(s)}\left[-\frac{d}{d\theta}f(\theta;s) - (\lambda-1)[1-sv^*(s)]g(\theta;s)\right]
\end{aligned} \tag{10}$$

where

$$g(\theta;s) = \frac{4}{\lambda-1}[-c_3(s)\cos(\lambda-1)\theta + c_4(s)\sin(\lambda-1)\theta]. \tag{11}$$

Substituting eqns (9) and (11) into eqn (10) and combining these equations with the transformed boundary conditions (6) and (7), one can get the following homogeneous system of four equations

$$\begin{aligned}
&-c_1(s)\sin\frac{\pi}{2}(\lambda+1) + c_2(s)\cos\frac{\pi}{2}(\lambda+1) - c_3(s)\sin\frac{\pi}{2}(\lambda-1) + c_4(s)\cos\frac{\pi}{2}(\lambda-1) = 0 \\
&c_1(s)(\lambda+1)\cos\frac{\pi}{2}(\lambda+1) + c_2(s)(\lambda+1)\sin\frac{\pi}{2}(\lambda+1) \\
&\qquad + c_3(s)(\lambda-1)\cos\frac{\pi}{2}(\lambda-1) + c_4(s)(\lambda-1)\sin\frac{\pi}{2}(\lambda-1) = 0 \\
&-c_2(s)(\lambda+1) + c_4(s)[-(\lambda+1) + 4(1-sv^*(s))] = 0 \\
&-c_1(s)(\lambda+1) + c_3(s)[-(\lambda-1) - 4(1-sv^*(s))] = 0.
\end{aligned} \tag{12}$$

A nontrivial solution to the equation exists only if the determinant of the coefficient matrix vanishes. This occurs when λ satisfies the following equation

$$\frac{2\lambda^2}{s} - 8s[v^*(s)]^2 + 12v^*(s) - \frac{5}{s} - \left[\frac{3}{s} - 4v^*(s)\right]\cos(\lambda\pi) = 0. \tag{13}$$

The time-dependent behavior of the problem is recovered by inverting eqn (13) into the real time space.

In order to examine the viscoelastic behavior at the interface corner of a viscoelastic adhesive layer bonded between rigid adherends, the viscoelastic model characterized by a standard solid shear relaxation modulus and a constant bulk modulus is taken as follows:

$$\mu(t) = g_0 + g_1 \exp\left(-\frac{t}{t^*}\right)$$
$$k(t) = k_0 \tag{14a}$$

where $\mu(t)$ is a shear relaxation modulus, $k(t)$ is a bulk modulus, g_0, g_1 and k_0 are positive constants, and t^* is the relaxation time. Clearly,

$$\mu(0) = g_0 + g_1$$
$$\mu(\infty) = g_0 < \mu(0). \tag{14b}$$

Introducing eqn (14a) into eqn (13) and rearranging the resulting equation, we have

$$\frac{2\lambda^2}{s} - 8A_1^*(s) + 12A_2^*(s) - \frac{5}{s} - \left[\frac{3}{s} - 4A_2^*(s)\right]\cos(\lambda\pi) = 0 \tag{15}$$

where

$$A_1^*(s) = \frac{1}{4}\left[\frac{3k_0 - 2\mu(0)}{3k_0 + \mu(0)}\right]^2 \frac{\left[s + \dfrac{3k_0 - 2\mu(\infty)}{3k_0 - 2\mu(0)}\dfrac{1}{t^*}\right]^2}{s\left[s + \dfrac{3k_0 + \mu(\infty)}{3k + \mu(0)}\dfrac{1}{t^*}\right]^2}$$

$$A_2^*(s) = \frac{[3k_0 - 2\mu(0)]}{2[3k_0 + \mu(0)]} \frac{\left[s + \dfrac{3k_0 - 2\mu(\infty)}{3k_0 - 2\mu(0)}\dfrac{1}{t^*}\right]}{s\left[s + \dfrac{3k_0 + \mu(\infty)}{3k + \mu(0)}\dfrac{1}{t^*}\right]}. \tag{16}$$

Equation (15) can be inverted analytically as follows:

$$2\lambda^2 - 8A_1(t) + 12A_2(t) - 5 - [3 - 4A_2(t)]\cos(\lambda\pi) = 0 \tag{17}$$

where

$$A_1(t) = \frac{1}{4}\left[\frac{3k_0 - 2\mu(0)}{3k_0 + \mu(0)}\right]^2 \left[\beta_1^2 + \left(1 - \beta_1^2 + \beta_2\frac{t}{t^*}\right)\exp\left(-\gamma\frac{t}{t^*}\right)\right]$$

$$A_2(t) = \frac{[3k_0 - 2\mu(0)]}{2[3k_0 + \mu(0)]}\left[\beta_1 + (1 - \beta_1)\exp\left(-\gamma\frac{t}{t^*}\right)\right] \tag{18}$$

and

$$\beta_1 = \frac{[3k_0+\mu(0)][3k_0-2\mu(\infty)]}{[3k_0+\mu(\infty)][3k_0-2\mu(0)]}$$

$$\beta_2 = 2\frac{3k_0-2\mu(\infty)}{3k_0-2\mu(0)} - \frac{3k_0+\mu(\infty)}{3k_0+\mu(0)} - \frac{3k_0+\mu(0)}{3k_0+\mu(\infty)}\left[\frac{3k_0-2\mu(\infty)}{3k_0-2\mu(0)}\right]^2$$

$$\gamma = \frac{3k_0+\mu(\infty)}{3k_0+\mu(0)}. \tag{19}$$

It can be easily verified that eqn (17) for $t = 0$ and $t \to \infty$ is written as follows:
for $t = 0$

$$2\lambda^2 - 8[v(0)]^2 + 12v(0) - 5 - [3-4v(0)]\cos(\lambda\pi) = 0 \tag{20}$$

for $t \to \infty$

$$2\lambda^2 - 8[v(\infty)]^2 + 12v(\infty) - 5 - [3-4v(\infty)]\cos(\lambda\pi) = 0. \tag{21}$$

Equations (20) and (21) have a form identical with that of an elastic adhesive layer bonded between rigid adherends and are equivalent to that reported by Bogy (1968). The singularity at the interface corner has a form of $r^{1-\lambda}$. Roots of eqn (17) with $0 < Re(\lambda) < 1$ are of main interest. The calculation of the zeros of eqn (17) must be carried out numerically for given values of material properties. For $0 < v(t) < 0.5$, there is at most one root λ_1 with $0 < Re(\lambda) < 1$, and that root is real. A more detailed discussion of the root of eqn (17) is presented in Bogy (1968).

In some cases the order of the singularity is independent of time. This situation can be provided by considering the following viscoelastic model:

$$v(t) = v_0$$
$$\mu(t) = g_0 + g_1 \exp\left(-\frac{t}{t^*}\right) \tag{22}$$

where v_0 is a constant Poisson's ratio. Substituting eqn (22) into eqn (13) and rearranging the resulting equation, we have

$$2\lambda^2 - 8[v_0]^2 + 12v_0 - 5 - [3-4v_0]\cos(\lambda\pi) = 0. \tag{23}$$

As is evident from eqn (23), choosing a constant Poisson's ratio provides a situation in which the order of the singularity is independent of time.

3. BOUNDARY ELEMENT SOLUTION FOR THE INTENSITY OF STRESS SINGULARITY

Figure 2(a) shows an idealized configuration that models a viscoelastic adhesive layer bonded between rigid adherends, subjected to a uniform transverse tensile strain $\varepsilon_0 H(t)$. Here $H(t)$ represents Heaviside unit step function. The adhesive and the adherend are considered to be perfectly bonded, with no defects or cracks. The layer has thickness $2h$ and length $2L$. Due to the symmetry, only one quarter of the layer needs to be modeled. Figure 2(b) represents the two-dimensional plane strain model for analysis of the stresses which develop at the interface between the adhesive and the adherend. Calculations are performed for $L/h = 25$.

Assuming that no body forces exist, the boundary integral equations for the analysis model can be written as follows:

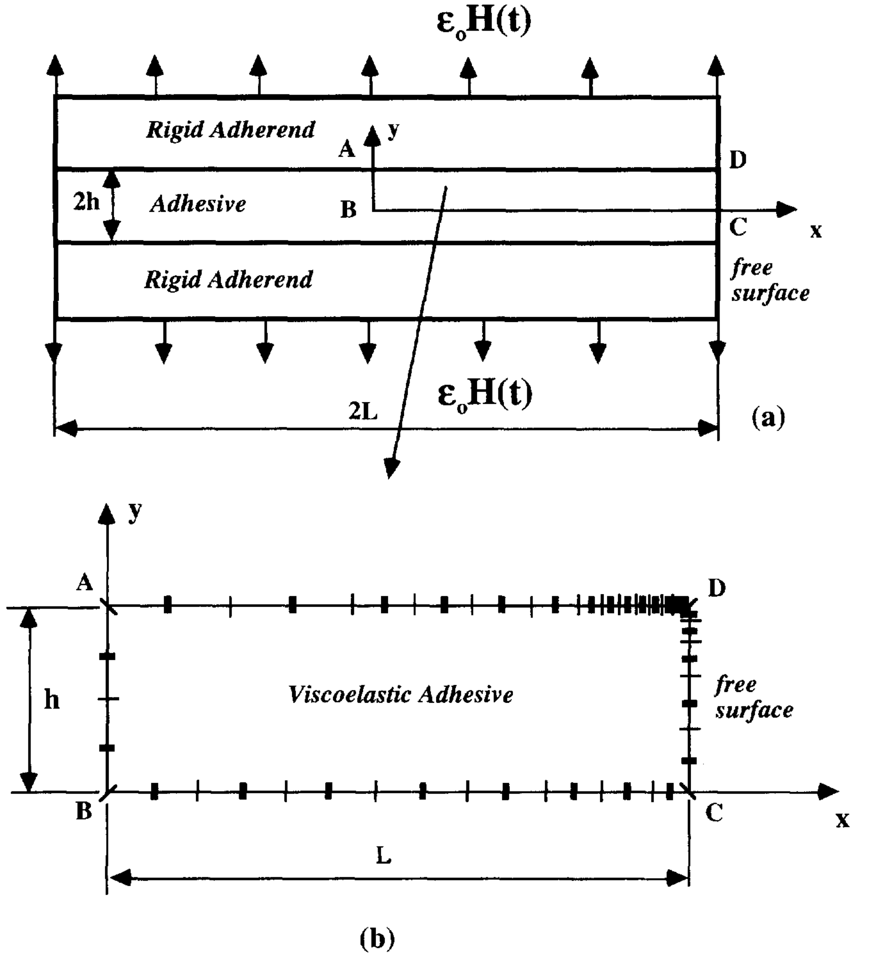

Fig. 2. Analysis model for determination of interface stresses developed in the viscoelastic adhesive layer.

$$c_{ij}(\mathbf{y})u_j(\mathbf{y},t)+\int_S\left[u_j(\mathbf{y}',t)T_{ij}(\mathbf{y},\mathbf{y}';0+)+\int_{0+}^{t}u_j(\mathbf{y}',t-t')\frac{\partial T_{ij}(\mathbf{y},\mathbf{y}';t')}{\partial t'}\mathrm{d}t'\right]\mathrm{d}S(\mathbf{y}')$$
$$=\int_S\left[t_j(\mathbf{y}',t)U_{ij}(\mathbf{y},\mathbf{y}';0+)+\int_{0+}^{t}t_j(\mathbf{y}',t-t')\frac{\partial U_{ij}(\mathbf{y},\mathbf{y}';t')}{\partial t'}\mathrm{d}t'\right]\mathrm{d}S(\mathbf{y}') \quad (24)$$

where u_j and t_j denote displacements and tractions, and S is the boundary of the given domain. The arguments $(\mathbf{y}, t)$ imply that the variables are dependent upon both the position $\mathbf{y}$ and the time t. $c_{ij}(\mathbf{y})$ is dependent only upon the local geometry of the boundary. For $\mathbf{y}$ on a smooth surface, the free-term $c_{ij}(\mathbf{y})$ is simply a diagonal matrix 0.5 δ_{ij}. The viscoelastic fundamental solutions U_{ij} and T_{ij} can be obtained by applying the elastic–viscoelastic correspondence principle to the elastic fundamental solutions.

Closed-form integrations of eqn (24) are not, in general, possible and, therefore, numerical quadrature must be used. Approximations are required in both time and space. In this study, eqn (24) is solved in a step-by-step fashion in time by using the modified Simpson's rule for the time integrals and employing the standard BEM for the surface integrals. The detailed calculation procedure for eqn (24) is provided in Lee and Westmann (1995). The resulting system of equations is obtained in the matrix form as follows:

$$[\mathbf{H}]\{\mathbf{u}\} = [\mathbf{G}]\{\mathbf{t}\}+\{\mathbf{R}\}. \quad (25)$$

In eqn (25), **H** and **G** are influence matrices and **R** is the hereditary effect due to the viscoelastic history. The above eqn (25) can be solved by taking account of the external boundary conditions. The resulting boundary conditions for the analysis model are given as follows:

$$
\begin{aligned}
\tau_{xy} &= 0, \quad u_x = 0 \quad \text{along A–B} \\
\tau_{xy} &= 0, \quad u_y = 0 \quad \text{along B–C} \\
\tau_{xy} &= 0, \quad \sigma_{xx} = 0 \quad \text{along C–D} \\
u_x &= 0, \quad u_y = h\varepsilon_0 \quad \text{along D–A.}
\end{aligned}
\tag{26}
$$

Applying the above boundary conditions to eqn (25) and solving the final system of equations at each time-step lead to determination of all boundary displacements and tractions.

In order to examine the viscoelastic behavior along the interface line of the analysis model subjected to a transverse tensile strain $\varepsilon_0 H(t)$, the viscoelastic model characterized by eqn (14a) is employed. The numerical values used in this example are as follows:

$$
\begin{aligned}
\mu(0) &= 0.55\times 10^3 \text{ MPa} \\
\mu(\infty) &= 0.11\times 10^3 \text{ MPa} \\
k_0 &= 2.0\times 10^3 \text{ MPa} \\
t^* &= 10 \text{ min} \\
\varepsilon_0 &= 0.01.
\end{aligned}
\tag{27}
$$

A suitable mesh density was determined for the analysis based upon the results of a convergence study for mesh refinement. The refined mesh was used near the interface corner. The boundary element discretization consisting of 23 line elements was employed. In this study, quadratic shape functions were used to describe both the geometry and functional variations. Viscoelastic stress profiles were plotted along interface to investigate the nature of stresses. Figure 3 shows the distribution of normal stress σ_{yy} and shear stress τ_{xy} on the interface at nondimensional times $t/t^* = 0$ and 10. The numerical results exhibit

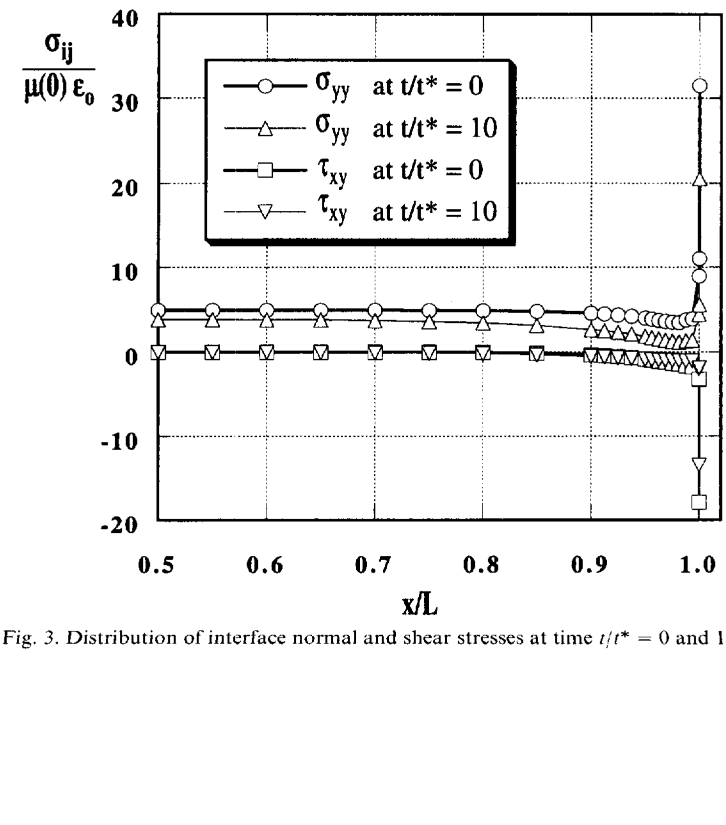

Fig. 3. Distribution of interface normal and shear stresses at time $t/t^* = 0$ and 10.

the relaxation of interface stresses and large gradients are observed in the vicinity of the free surface.

The singular stress levels near the free-edge can be characterized by two parameters: the order of the singularity and the free-edge stress intensity factor. The order of the singularity must be determined from the roots of the characteristic eqn (17). The free-edge stress intensity factor was defined first by Wang and Choi (1982). In this study, the free-edge intensity factor is normalized by the quantity $h^{1-\lambda}$, giving it stress units, as follows:

$$K_{ij} = \lim_{r\to 0}\left(\frac{r}{h}\right)^{1-\lambda}\sigma_{ij}(r,0;t). \tag{28}$$

Figure 4 shows the variation of the order of the singularity with time for the material properties given by eqn (27). Since the value of Poisson's ratio of the viscoelastic adhesive becomes greater with time, the order of the singularity increases with time. To check the accuracy of the results, it is interesting to consider an elastic case for the viscoelastic adhesive material with shear modulus $\mu(\infty)$: i.e. the viscoelastic adhesive layer of Fig. 2(b) is replaced by an elastic material with $\mu(\infty)$ and the rigid adherend remains unchanged. At greater times, the order of the singularity in Fig. 4 approaches that for the analysis model consisting of elastic adhesive with $\mu(\infty)$ and rigid adherend. Figure 5 shows the variation of the free-edge stress intensity factor. It is shown that the free-edge stress intensity factor is relaxed with time while the order of the singularity increases with time. It is, however, unclear how these competing effects will affect failure or adhesive–adherend debonding.

Figure 6 shows the results for the relaxation of the interface stresses in a case where the order of the singularity is independent of time. The viscoelastic model employed in this case is identical with that given in eqn (22). The numerical values used in this example are as follows:

$$\mu(0) = 10^3 \text{ MPa}$$
$$\mu(\infty) = 0.5\times 10^3 \text{ MPa}$$
$$v_0 = 0.35$$
$$t^* = 10 \text{ min}$$
$$\varepsilon_0 = 0.01. \tag{29}$$

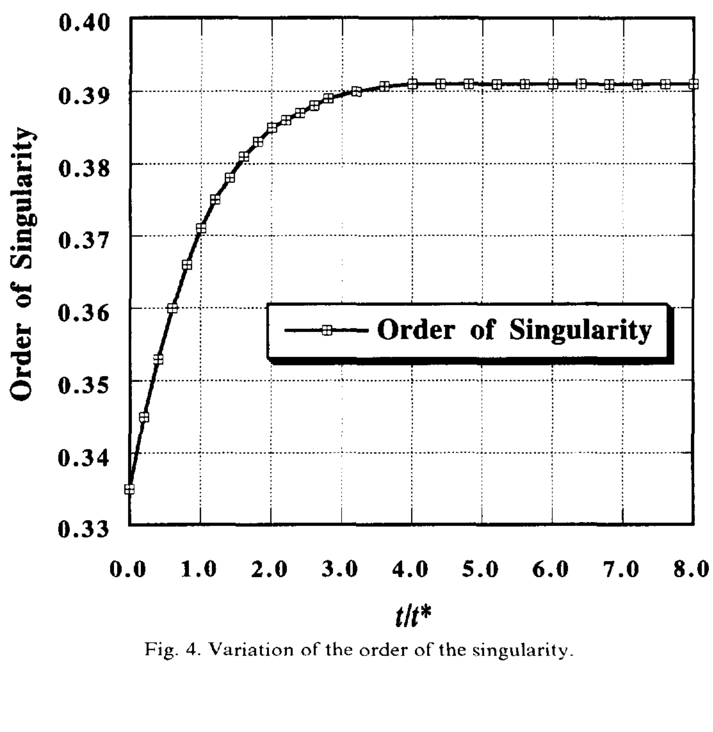

Fig. 4. Variation of the order of the singularity.

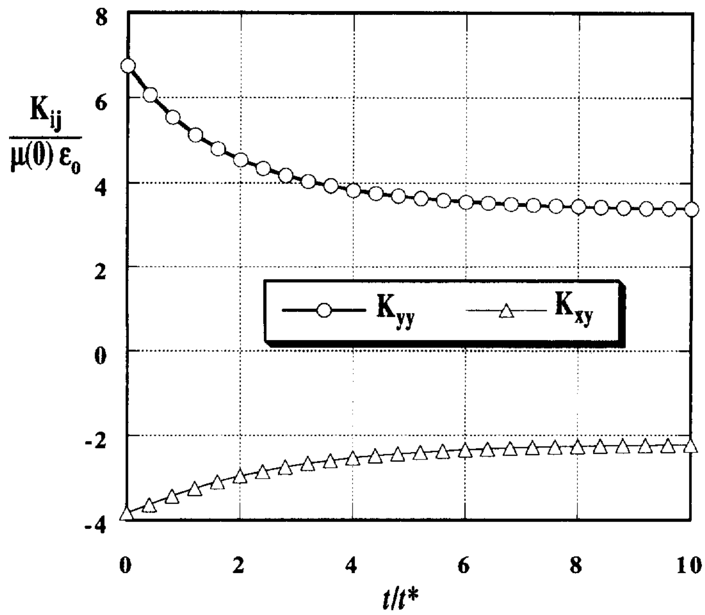

Fig. 5. Variation of the free-edge stress intensity factors.

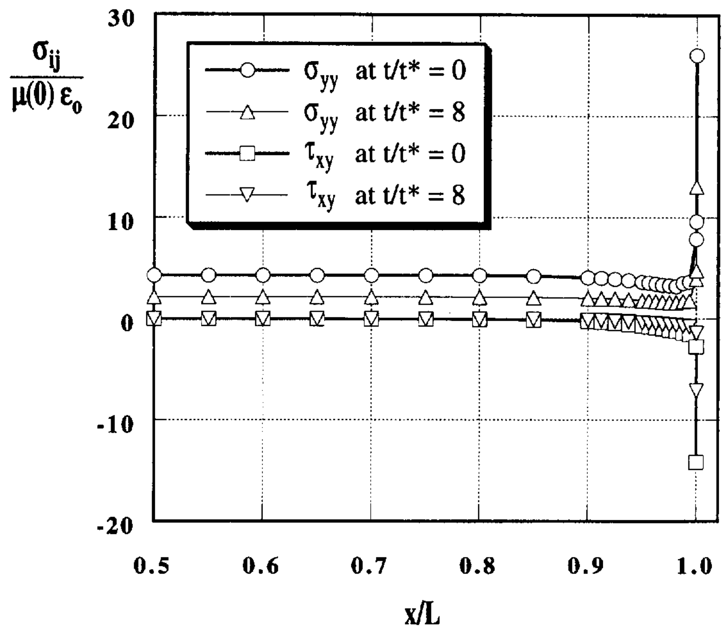

Fig. 6. Distribution of interface normal and shear stresses at time $t/t^* = 0$ and 8.

The order of the singularity for this case is 0.32. Figure 7 shows the decay of the stress intensity factor for a constant order of the singularity. For the viscoelastic models considered here, it is shown that the free-edge stress intensity factors are relaxed with time while the order of the singularity increases with time or remains constant.

4. CONCLUSIONS

The singular stresses at the interface corner between the viscoelastic adhesive layer and the rigid adherend subjected to a uniform transverse tensile strain have been investigated by using the time-domain boundary element method. Numerical results show that very large stress gradients are present at the interface corner and such stress singularity dominates a very small region relative to layer thickness. it is also shown that the free-edge stress

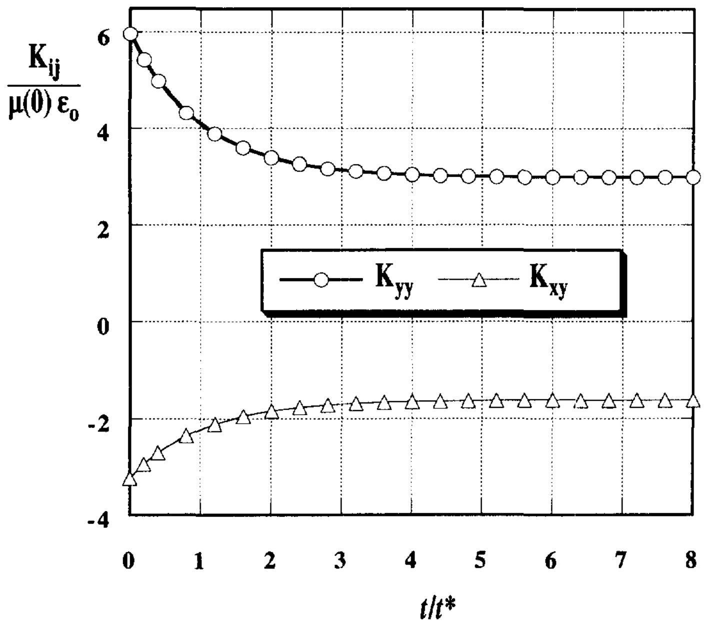

Fig. 7. Variation of the free-edge stress intensity factors.

intensity factor is relaxed with time while the order of the singularity increases with time or remains constant for viscoelastic models considered here. Since the exceedingly large stresses at the interface corner cannot be borne by adhesive layer, local yielding or adhesive–adherend debonding can occur in the vicinity of free surface.

REFERENCES

Bogy, D. B. (1968) Edge-bonded dissimilar orthogonal elastic wedges under normal and shear loading. *ASME Journal of Applied Mechanics* **35**, 460–466.

Delale, F. and Erdogan, F. (1981) Viscoelastic analysis of adhesively bonded joints. *ASME Journal of Applied Mechanics* **48**, 331–338.

Lee, S. S. and Westmann, R. A. (1995) Application of high-order quadrature rules to time-domain boundary element analysis of viscoelasticity. *International Journal for Numerical Methods in Engineering* **38**, 607–629.

Reedy, E. D. Jr (1990) Intensity of the stress singularity at the interface corner between a bonded elastic and rigid layer. *Engineering Fracture Mechanics* **36**, 575–583.

Tsai, M. Y. and Morton, J. (1991) The stresses in a thermally loaded bimaterial interface. *International Journal Solids and Structures* **28**, 1053–1075.

Wang, S. S. and Choi, I. (1982) Boundary layer effects in composite laminates: part 2—free edge stress solutions and basic characteristics. *ASME Journal of Applied Mechanics* **49**, 549–560.

Weitsman, Y. (1979) Interfacial stresses in viscoelastic adhesive-layers due to moisture sorption. *International Journal of Solids and Structures* **15**, 701–713.

Williams, M. L. (1952) Stress singularities resulting from various boundary conditions in angular corners of plates in extension. *ASME Journal of Applied Mechanics* **74**, 526–528.